2018-00674 - [CORDIS2018-DATASHAPE] PhD Student - Algorithms and Complexity in Geometric Topology

Level of qualifications required: Graduate degree or equivalent
Function: PhD Position

About the research centre or Inria department

The Inria Sophia Antipolis - Méditerranée center counts 37 research teams and 9 support departments. The center's staff (about 600 people including 400 Inria employees) is composed of scientists of different nationalities (250 foreigners of 50 nationalities), engineers, technicians and administrators. 1/3 of the staff are civil servants, the others are contractual. The majority of the research teams at the center are located in Sophia Antipolis and Nice in the Alpes-Maritimes. Six teams are based in Montpellier and a team is hosted by the computer science department of the University of Bologna in Italy. The Center is a member of the University and Institution Community (ComUE) "Université Côte d’Azur (UCA)."

Context

The DataShape research group offers an excellent environment to conduct this research project. The team leader in SAM, Jean-Daniel Boissonnat, is an expert in computational geometry, and has in particular been working in the past few years in extending computational methods, such as Delaunay triangulations and meshing, to non-Euclidean spaces. Other main collaborator of the project, Clément Maria, is an expert in computational topology, and has in particular been working on algorithm and complexity for solving 3-manifold problems, using computer science approaches such as parameterised complexity. The DataShape group also offers possibilities for collaborations with Mathijs Wintraecken, specialised in computational Riemannian geometry, and Kunal Dutta, specialised in combinatorial geometry. At the international scale, computational low-dimensional topology is a young and very active field, and has for example been the focus of workshops at renowned international research centers in the recent years, such as Oberwolfach (MFO, 2015) and the Lorenz center (Leiden, 2015). The advisors are also collaborating with Benjamin Burton (the University of Queensland), Jessica S. Purcell (Monash University), Arnaud de Mesmay (GIPSA-lab), and Jonathan Spreer (FU Berlin) on related questions, which offers potential to collaborations to the PhD candidate. Part of this thesis is also to develop and release new software, which encourages communication with the SNAPPY community and the REGINA community.

Assignment

Low dimensional topology is the area of mathematics focusing on the study of knots, surfaces and 3-manifolds. Motivated by natural questions, such as the one of telling distinct knots apart, the field has been a driving force of modern topology in the XXth century, building complex tools and growing deep ramifications in algebra, theoretical physics, combinatorics and, more recently, computer science. One of the main breakthroughs in low dimensional topology was the introduction of geometric structures to study knots and manifolds, culminating in Thurston’s hyperbolisation theorem in the late 70th, and the proof of his geometrisation conjecture by Perelman in the early 2000. Essentially, and much like orientable surfaces can be split into collections of tori, these theorems state that certain 3-manifolds of interest can be cut canonically along spheres and tori into a collection of “elementary” pieces, that each admit one of eight types of geometric structure. The focus of this project is mainly on one of the eight geometric structures of 3-manifolds mentioned above, known as the hyperbolic geometry. The hyperbolic structure is of strong interest for several reasons. First, it is the most prevalent geometric structure ; for example, most knots K admit a hyperbolic structure in their complement with the 3-sphere S3 - K, and this geometry is extremely powerful to tell distinct knots apart. Second, the hyperbolic structure is the least understood mathematically, which asks for automated methods to study it. Computations in geometric topology. Interestingly for a field dealing with continuous spaces and metrics, a lot of mathematical progress has been made through the use of discrete structures to represent 3-manifolds -- such as (generalised/ideal) triangulations --, and their hyperbolic geometry -- encoded with angle structures --, as well as the development of powerful algorithmic techniques, like Haken’s normal surface theory. Moreover, computations play a major role in the systematic study of knots and manifolds, in particular with the creation of knots and 3-manifold tables, and the use of powerful software, such as SNAPPY for hyperbolic knots, and REGINA and the MANIFOLD RECOGNIZER for 3-manifolds, to prove theorems and elaborate conjectures in low dimensional topology. In the past couple of decades, theoretical computer science has entered the field of low dimensional topology, with the analysis of the classical, quantum and parameterised computational complexity of various problems, and the
design and implementation of efficient practical algorithms. However, little is known on the complexity of problems related to geometric topology, and provably correct algorithms with well-analysed complexity are few. In parallel, the field of computational geometry in computer science has witnessed progress towards the study of non-linear geometric spaces, such as triangulated Riemannian manifolds, for which hyperbolic manifolds are a particular case. As mentioned above, geometry and topology work hand in hand in hyperbolic manifolds. By using recent techniques in computational geometry and computational topology, and exploiting the connection between hyperbolic structures and the topology of manifolds, the aim of the thesis is: Overarching Aim: to study the algorithmic complexity of problems related to the hyperbolic geometry of knots and 3-manifolds, to design new efficient and provably correct algorithms to solve them, and to develop and release quality open source software to be used by the mathematics community. Bibliography: - a full bibliography for the context of this thesis can be found at: `http://www-sop.inria.fr/members/Clement.Maria/hypgeom.pdf` - a full bibliography for the context of this thesis can be found at `http://www-sop.inria.fr/members/Clement.Maria/hypgeom.pdf`

**Main activities**

We distinguish the following directions of research, with short term questions and long term problems, in the following sections. 1/ Computational complexity of hyperbolic structures in our setting. A 3-manifold $M$ is represented by a triangulation $T$, made of a collection of $n$ abstract tetrahedra, together with a face pairing relation "gluing" triangular faces in pairs. We will first focus on ideal triangulations with toric boundary, which are ubiquitous in geometric topology and allow to represent non-compact 3-manifold such as knot complements. They additionally admit a unique, up to isometry, hyperbolic geometry. Based on works of Thurston, and Rivin and Casson, a complete hyperbolic structure on $M$ is represented by an angle structure, which assigns to every dihedral angle of $T$ a complex number. These angles encode a "hyperbolic shape" for each tetrahedron and gives a local embedding in hyperbolic space. When the angles satisfy a set of "coherence" equations, the angle structure encodes a full complete hyperbolic metric on the manifold. We plan to study the computational complexity of, and efficient algorithms for, the following problems: Problem 1: Given a triangulation $T$ of a 3-manifold $M$, decide whether or not it admits an angle structure. Problem 2: Given a triangulation $T$ of a 3-manifold $M$, compute explicitly, if it exists, an angle structure. For the computational hardness of Problem 1, a first line of research will be to explore the possibility of generalising [Burton and Spreer 13]'s NP-completeness proof of finding taut angle structures, a very restricted family of angle structures. The proof of hardness relies on a reduction from a version of SAT by designing, using computation and the manifold census, "pieces of manifolds" as reduction gadgets, and combining them together. We plan to generalise the approach. We will also design efficient algorithms in various complexity models. Problem 1 can be reduced to deciding whether a convex polytope $A(T)$, defined by the input triangulation $T$, is empty [Futer, Guéritaud 10]. We will study the decision Problem 1 along this direction, specialised in the case of efficient triangulations [Jaco, Rubinstein 03], which have nice combinatorics and topology. In order to solve Problem 2, we will also explore the possibility of a polynomial time approximation algorithm for approximating angle structures, and analyse complexities in the parameterised complexity model. Finally, it is known that, under mild conditions, a hyperbolic manifold $M$ admits an ideal triangulation with an angle structure, but not all triangulations of $M$ admit one. This rises naturally the following combinatorial problem: Problem 3: Given a triangulation $T$ of a hyperbolic manifold $M$, find a triangulation $T'$ of $M$ admitting an angle structure. and refinements of the problem, such as the one of finding a triangulation admitting an angle structure whose angles have bounded arithmetic complexity. A direction of research to solve Problem 3 is to study Pachner moves to connect triangulations of a same manifold, to study the evolution of angle structures under such transformations, and to find combinatorial obstruction in the triangulation to the existence of angle structures. 2/ Computational geometry in triangulated hyperbolic manifolds A second line of research for this project is to solve algorithmic problems on triangulations of hyperbolic manifolds equipped with angle structures, encoding their metric. It strongly relies on the use of computational geometry tools in non-Euclidean spaces. To the best of our knowledge, this line of research as very little been studied yet. By Margulis' lemma, a key feature of hyperbolic manifolds is their "topologically nice" thick-thin decomposition, that consists in splitting a hyperbolic manifold $M$ into a thin part -- made of "short geodesics" -- and a thick part -- its complement. Consequently, we plan to study the following computational problem: Problem 4: Given a triangulation $T$ of a hyperbolic manifold $M$, with an angle structure, find a (closed) triangulation $T'$ of a thin part of $M$ and an (ideal) triangulation $T_2$ of the thin part of $M$, together with a gluing of $T_1$ and $T_2$ to recover a triangulation of $M$. Solving this problem will require to formalise several notions of computational geometry to the setting of 3-manifold triangulations equipped with angle structures; in particular: - computation of hyperbolic distances, geodesics, and higher-order geometric predicates (with exact or approximate arithmetic), - meshing, and geometric subdivision of the triangulation and its angle structure. The complexity of an algorithm to solve Problem 4 should depend on the number of tetrahedra of the input triangulation $T$, and the hyperbolic volume of the manifold $M$. Finding such decomposition has strong applications in the study of hyperbolic manifold, and in simplifying their triangulations. Other directions of research will include the study of the computation of other decomposition in this framework, including Heegaard splittings. Collaboration: The PhD student will closely work within the DATASHAPE research project of the Sophia-Antipolis Inria Research Center.

**Skills**

Master degree in Computer Science or Mathematics, or equivalent, is required. Programming skills. Knowledge of complexity theory and algorithms is preferred. Fluent English required, both oral and written. Knowledge of French is not required.
**Benefits package**
- Subsidised catering service
- Partially-reimbursed public transport
- Social security
- Paid leave
- Flexible working hours
- Sports facilities

**Remuneration**
Duration: 36 months
Location: Sophia Antipolis, France
Gross Salary per month: 1982€ brut per month (year 1 & 2) and 2085€ brut/month (year 3)